Quantum scattering from the Coulomb potential plus an angle-dependent potential: a group-theoretical study

2007 J. Phys. A: Math. Theor. 40 7297

(http://iopscience.iop.org/1751-8121/40/26/014)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 7297–7308 doi:10.1088/1751-8113/40/26/014

Quantum scattering from the Coulomb potential plus an angle-dependent potential: a group-theoretical study

G A Kerimov

International Centre for Physics and Applied Mathematics, Trakya University, 22050 Edirne, Turkey

Received 6 December 2006, in final form 9 May 2007
Published 12 June 2007
Online at stacks.iop.org/JPhysA/40/7297

Abstract
The nonrelativistic quantum scattering problem for a non-central potential which belongs to a class of potentials exhibiting an 'accidental' degeneracy is studied. We show that the scattering system under consideration admit the Lie algebra $\mathfrak{so}(5, 1)$ as the potential algebra. The scattering amplitude is then evaluated using purely algebraic techniques to give the closed result. It is expressed in terms of associated Legendre functions.

PACS numbers: 03.65.Nk, 02.20.−a

1. Introduction

The occurrence of 'accidental' degeneracy in the energy spectrum (i.e. a degeneracy of the energy levels not connected with obvious geometric symmetries of the Hamiltonian) has caught the attention of many physicists since the pioneering paper [1] appeared. Many examples of potentials with accidental degeneracy in the energy spectrum are now known; the paper [2] lists a variety of potentials leading to accidental degeneracy. It turns out that they possess properties making them of special interest; for instance, all these potentials admit the separation of variables in several coordinate systems and possess dynamical symmetries responsible for separability of the Schrödinger equation. Moreover the 'accidental' degeneracy occurring in these problems have been explained in terms of dynamical symmetries. The best known of these potentials are Coulomb [1], oscillator [3] and Hartmann potentials [4]. The latter results from adding a repulsive potential proportional to $1/r^2 \sin^2\theta$ to the Coulomb one. It was first studied by Hartmann to describe axial symmetric systems such as ring-shaped molecules.

The potentials classified in [2] involve potentials that may be written in spherical coordinates as $f(\theta,\varphi)/r^2 + V(r)$, where $f(\theta,\varphi)$ are certain functions of θ and φ. They are exactly solvable if $V(r)$ are Coulomb, harmonic oscillator or null potentials. (By exactly solvable, one means those Hamiltonians for which the spectrum, eigenfunctions and the scattering matrices can be found explicitly.) This has been treated using a path integral approach [5–12], supersymmetric quantum mechanics [13, 14], a group theoretical approach [15–27], as well as, the standard approach [4, 28–32]. Apart from their beautiful mathematical structure, these solvable potentials provide excellent models in quantum chemistry and nuclear physics for describing ring-shaped molecules and deformed nuclei. What is still lacking is an explicit description of scattering problems for these non-central potentials. Despite several attempts at the problem [16, 12, 31], at present, the potential $a/r + c/r^2(1 - \cos\theta)$ is the only example of a scattering amplitude which has been calculated [16]. However, in this study, the authors restrict themselves to the case when the incident wave is along the z-axis.

In [33], we proposed a way which allows pure algebraic calculation of S-matrices for the systems whose Hamiltonians are related to the Casimir operators C_i of some Lie group G

$$H = f(C_i) \tag{1}$$

or

$$H = f(C_i)|_{\mathfrak{H}} \tag{2}$$

where $\mathfrak{H}$ is some subspace of carrier space and $|_{\mathfrak{H}}$ denotes the restriction to $\mathfrak{H}$. (In the later case the group G describes fixed energy states of a family of quantum systems with different potential strength and therefore is called the 'potential' group [34].) Namely, the S-matrices for the systems under consideration are associated with intertwining operators A between Weyl equivalent representations U^{χ} and $U^{\tilde{\chi}}$ of G

$$S = A \tag{3}$$

or

$$S = A|_{\mathfrak{H}} \tag{4}$$

respectively. (The representations U and $U^{\tilde{\chi}}$ have the same Casimir eigenvalues. Such representations are called Weyl equivalent.) At this stage we note that the operator A is said to intertwine the representations U^{χ} and $U^{\tilde{\chi}}$ of the group G if relation

$$AU^{\chi}(g) = U^{\tilde{\chi}}(g)A \qquad \text{for all} \quad g \in G \tag{5}$$

or

$$AdU^{\chi}(b) = \mathrm{d}U^{\tilde{\chi}}(b)A \qquad \text{for all} \quad b \in \mathfrak{g} \tag{6}$$

holds, where $\mathrm{d}U^{\chi}$ and $\mathrm{d}U^{\tilde{\chi}}$ are the corresponding representations of the algebra $\mathfrak{g}$ of G. Equations (5) and (6) have much restriction power, determining the intertwining operator up to a constant. Therefore, one can evaluate the S-matrix without writing wavefunctions. Such approaches were applied to various classes of solvable potentials [35–40]. Among the class of problems which could be treated using the technique discussed above, there are the well-known Coulomb and MICZ-Kepler [41, 42] problems (see [39]).

Here we provide an algebraic approach to the scattering problem for the non-spherically symmetric potentials

$$V(\mathbf{x}) = -\frac{\gamma}{r} + a_0^2\varepsilon_0\frac{s(s+1)}{r^2\sin^2\theta\sin^2\varphi} \tag{7}$$

which appears among the potentials classified in [2]. Here r, θ, φ are spherical coordinates, and a_0 and ε_0 stand for the Bohr radius and the ground-state energy of the hydrogen atom, respectively. We show that the potentials (7) admit the Lie algebra $\mathfrak{so}(5, 1)$ as the potential algebra

$$H = -\frac{\gamma^2}{2(C+4)}\bigg|_{\mathfrak{H}_s},$$

where C is the second-order Casimir operator of $\mathfrak{so}(5,1)$, while $\mathfrak{H}_s$ is a subspace occurring in the subalgebra reduction $\mathfrak{so}(5,1) \supset \mathfrak{so}(5) \supset \cdots \supset \mathfrak{so}(2)$.

This paper could be considered as a natural continuation of the study started in [27], where bound-state problems for such potentials were presented. It should be noted that the strength of the non-central part in that paper was proportional to $n^2 - 1/4$, $n = 0, 1, 2, \ldots$, while in the present paper it is proportional to $s(s+1)$, $s = 0, 1, 2, \ldots$. That is why we come to the algebra $\mathfrak{so}(5,1)$ instead of $\mathfrak{so}(4,1)$. (The parameter $s(s+1)$ makes it possible to obtain from (7) the Coulomb potential.)

2. $\mathfrak{so}$(5,1) as a potential algebra

To describe the matter, we require some notation. By $SO(5,1)$ we denote the connected component of the group of linear transformations of a six-dimensional Minkowskian space $R^{5,1}$ preserving the bilinear form

$$[\xi, \eta] = \xi_1\eta_1 + \cdots + \xi_5\eta_5 - \xi_6\eta_6. \tag{8}$$

Let $\{g_{\mu\nu}(\theta)\}$ $(\mu < \nu;\ \mu, \nu = 1, 2, \ldots, 6)$ be the one-parameter subgroups of $SO(5,1)$ consisting of rotations or pseudo-rotations in the ξ_μ–ξ_ν planes. Then the matrices

$$a_{\mu\nu} = \frac{\mathrm{d}}{\mathrm{d}\theta} g_{\mu\nu}(\theta) \tag{9}$$

form a basis of Lie algebra $so(5,1)$ with commutation relations

$$\begin{aligned} &[a_{ij}, a_{kl}] = \delta_{ik}a_{jl} + \delta_{jl}a_{ik} - \delta_{il}a_{jk} - \delta_{jk}a_{il}, \\ &[a_{i6}, a_{j6}] = a_{ij}, \qquad i, j = 1, \ldots, 5, \\ &[a_{ij}, a_{k6}] = \delta_{ik}a_{j6} - \delta_{jk}a_{i6}. \end{aligned} \tag{10}$$

The generators a_{ij} $(1 \leqslant i < j \leqslant 5)$ form a Lie algebra $\mathfrak{so}(5)$ of $SO(5)$.

It is well known that the principal most-degenerate (or, class 1) unitary irreducible representation (UIR) of $\mathfrak{so}(5,1)$ can be realized (see, e.g., [43] and references therein) in the Hilbert space $\mathfrak{H}$ spanned by scattering states of the Coulomb Hamiltonian h in five dimensions, where

$$h = \frac{p^2}{2} - \frac{\gamma}{r}, \qquad \gamma > 0 \tag{11}$$

with $x = (x_1, x_2, \ldots, x_5)$, $p = (p_1, p_2, \ldots, p_5)$ and

$$r^2 = \sum_{i=1}^{5} x_i^2, \qquad p_i = -\mathrm{i}\frac{\partial}{\partial x_i}, \qquad i = 1, 2, \ldots, 5. \tag{12}$$

(We are using units with $M = \hbar = 1$.) As a prelude to this realization one introduces angular momentum and Runge–Lenz operators given by

$$L_{ij} = x_i p_j - x_j p_i, \tag{13}$$

$$A_i = \frac{1}{2}(L_{ij}p_j + p_j L_{ij}) - \frac{\gamma x_i}{r}. \tag{14}$$

These operators satisfy the following commutation relations:

$$[L_{ij}, L_{kl}] = \mathrm{i}(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il}), \tag{15}$$

$$[L_{ij}, A_k] = \mathrm{i}(\delta_{ik}A_j - \delta_{jk}A_i), \tag{16}$$

$$[A_i, A_j] = -2\mathrm{i}hL_{ij}, \tag{17}$$

$$[L_{ij}, h] = [A_i, h] = 0. \tag{18}$$

Defining now operators

$$L_{i6} = -L_{6i} \equiv \left(\frac{1}{2h}\right)^{1/2} A_i, \tag{19}$$

which are well defined in $\mathfrak{H}$ we obtain for $L_{\alpha\beta}, \alpha, \beta = 1, 2, \ldots, 6$ the commutation relations of the Lie algebra $\mathfrak{so}(5, 1)$

$$[L_{ij}, L_{kl}] = \mathrm{i}(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il}), \tag{20}$$

$$[L_{i6}, L_{j6}] = -\mathrm{i}L_{ij}, \tag{21}$$

$$[L_{ij}, L_{k6}] = \mathrm{i}(\delta_{ik}L_{j6} - \delta_{jk}L_{i6}). \tag{22}$$

Thus the most-degenerate UIR of $\mathfrak{so}(5, 1)$ is realized in the Hilbert space $\mathfrak{H}$ of the scattering wavefunctions $\Phi(x)$ corresponding to the fixed energy subspace, with inner product

$$(\Phi_1, \Phi_2)_{\mathfrak{H}} = \int_{R^5} \Phi_1^*(x)\Phi_2(x)\,\mathrm{d}^5x, \tag{23}$$

where $d^5x = \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_5$. In this realization, the representation operators are given by equations (13) and (14). If we compute the second-order Casimir operator

$$C = \sum_i L_{i6}^2 - \sum_{i<j} L_{ij}^2 \tag{24}$$

for this realization, it becomes

$$C = -4 - \frac{\gamma^2}{2h}. \tag{25}$$

We are looking for the chain $\mathfrak{so}(5, 1) \supset \mathfrak{so}(5) \supset \cdots \supset \mathfrak{so}(2)$. According to this the reduction conditions are

$$C^{\mathfrak{so}(5)}\,|\lambda msk\rangle = \lambda(\lambda + 3)\,|\lambda msk\rangle\,, \tag{26}$$

$$C^{\mathfrak{so}(4)}\,|\lambda msk\rangle = m(m + 2)\,|\lambda msk\rangle\,, \tag{27}$$

$$C^{\mathfrak{so}(3)}\,|\lambda msk\rangle = s(s + 1)\,|\lambda msk\rangle\,, \tag{28}$$

$$L_{12}\,|\lambda msk\rangle = k\,|\lambda msk\rangle\,, \tag{29}$$

where

$$C^{\mathfrak{so}(5)} = \frac{1}{2}\sum_{i,j=1}^{5} L_{ij}^2, \qquad C^{\mathfrak{so}(4)} = \frac{1}{2}\sum_{i,j=1}^{4} L_{ij}^2, \qquad C^{\mathfrak{so}(3)} = \frac{1}{2}\sum_{i,j=1}^{3} L_{ij}^2. \tag{30}$$

The parametrization that we seek for $x_1, x_2, \ldots, x_5$ must be such as to make $C^{\mathfrak{so}(5)}, C^{\mathfrak{so}(4)}, C^{\mathfrak{so}(3)}$ and L_{12} particularly simple. We define them as follows

$$x_1 = r\sin\theta\sin\varphi\sin\alpha\sin\beta,$$
$$x_2 = r\sin\theta\sin\varphi\sin\alpha\cos\beta,$$
$$x_3 = r\sin\theta\cos\varphi\cos\alpha,$$
$$x_4 = r\sin\theta\cos\varphi,$$
$$x_5 = r\cos\theta$$

with $0 \leqslant \alpha, \theta, \varphi < \pi, 0 \leqslant \beta < 2\pi$ and

$$d^5x = r^4 \sin^3\theta \sin^2\varphi \sin\alpha \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}\alpha \, \mathrm{d}\beta. \tag{31}$$

Then

$$C^{\mathfrak{so}(5)} = \frac{1}{\sin^3\theta}\frac{\partial}{\partial\theta}\sin^3\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{1}{\sin^2\varphi}\frac{\partial}{\partial\varphi}\sin^2\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\varphi\sin^2\alpha}\frac{\partial}{\partial\alpha}\sin^2\alpha\frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\varphi\sin^2\alpha}\frac{\partial^2}{\partial\beta^2}\right),$$

$$C^{\mathfrak{so}(4)} = -\left(\frac{1}{\sin^2\varphi}\frac{\partial}{\partial\varphi}\sin^2\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta\sin\alpha}\frac{\partial}{\partial\alpha}\sin\alpha\frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\theta\sin^2\alpha}\frac{\partial^2}{\partial\beta^2}\right),$$

$$C^{\mathfrak{so}(3)} = -\left(\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\sin\alpha\frac{\partial}{\partial\alpha} + \frac{1}{\sin^2}\frac{\partial^2}{\partial\beta^2}\right) \qquad \text{and} \qquad L_{12} = \mathrm{i}\frac{\partial}{\partial\beta},$$

while

$$\frac{\gamma^2}{(C+4)} = \frac{1}{r^4}\frac{\partial}{\partial r}r^4\frac{\partial}{\partial r} + \frac{1}{r^2}C^{\mathfrak{so}(5)} + \frac{2\gamma}{r}. \tag{32}$$

At this stage we note that, in general, one can use for the construction of the principal most-degenerate UIR of $\mathfrak{so}(5, 1)$ the carrier space with any quasi-invariant measure $\mathrm{d}\mu(x)$ on R^5. The representations with different measure are unitarily equivalent. Although the representations with different measure are equivalent from the mathematical viewpoint, they may be related to different physical problems (see below).

It is clear that we must construct the representation in the Hilbert space $\mathfrak{H}'$, with inner product

$$(\Phi_1', \Phi_2')_{\mathfrak{H}'} = \int_{R^5} \Phi_1'^*(x)\Phi_2'(x) \, \mathrm{d}\mu(x), \tag{33}$$

where

$$\mathrm{d}\mu(x) = r^2 \sin\theta \sin\alpha \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}\alpha \, \mathrm{d}\beta. \tag{34}$$

This representation, of course, is unitarily equivalent to the representation constructed in $\mathfrak{H}$. The unitary mapping W which realizes the equivalence is given by

$$W : \Phi \to \Phi' = (r \sin\theta \sin\varphi) \circ \Phi. \tag{35}$$

where $\circ$ denotes composition of operators. In this case, the generators, denoted as $L'_{\alpha\beta}, \alpha, \beta = 1, 2, \ldots, 6$, are given by

$$L'_{\alpha\beta} = (r \sin\theta \sin\varphi) \circ L_{\alpha\beta} \circ (r \sin\theta \sin\varphi)^{-1}.$$

That is, for the representation constructed in $\mathfrak{H}'$ the Casimir operator, call it C', is obtained by

$$C' = (r \sin\theta \sin\varphi) \circ C \circ (r \sin\theta \sin\varphi)^{-1}. \tag{36}$$

Hence

$$\frac{\gamma^2}{(C'+4)} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{1}{r^2\sin^2\theta\sin^2\varphi}\left(\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\sin\alpha\frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\alpha}\frac{\partial}{\partial\beta}\right) + \frac{2\gamma}{r}. \tag{37}$$

Let $\mathfrak{H}_s$ be a subspace spanned by $|\lambda msk\rangle$ with fixed s and k. Then the operator (37) restricted to $\mathfrak{H}_s$ becomes a differential operator in r, θ, φ; it is found that

$$\left.\frac{\gamma^2}{(C'+4)}\right|_{\mathfrak{H}_s} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) - \frac{s(s+1)}{r^2\sin^2\theta\sin^2\varphi} + \frac{2\gamma}{r}. \tag{38}$$

Hence the Hamiltonian

$$H = -\frac{1}{2}\nabla^2 - \frac{\gamma}{r} + \frac{s(s+1)}{2r^2\sin^2\theta\sin^2\varphi}, \qquad s = 0, 1, 2, \ldots \tag{39}$$

is related to $\mathfrak{so}(5,1)$ in the sense that the following relation holds

$$H = -\left.\frac{\gamma^2}{2(C'+4)}\right|_{\mathfrak{H}_s}. \tag{40}$$

As was mentioned before representations with different measure may be related to different physical problems. We see that the principal most-degenerate UIR of $\mathfrak{so}(5,1)$ realized in the Hilbert space $\mathfrak{H}'$ with the measure (34) provide descriptions of scattering states for Hamiltonian (39), whereas Coulomb scattering in five dimensions is described by the principal most-degenerate UIR of $\mathfrak{so}(5,1)$ realized in the Hilbert space $\mathfrak{H}$ with the measure (31). These results also establish a correspondence between Coulomb problem in five dimensions and the problem governed by Hamiltonian (39) in three dimensions. So, the scattering amplitude for (39) can also be obtained from the Coulomb amplitude.

The operators commuting with H are

$$\widetilde{L^2} = \mathbf{L}^2 + \frac{s(s+1)}{\sin^2\theta\sin^2\varphi} \tag{41}$$

and

$$\widetilde{L_z^2} = L_z^2 + \frac{s(s+1)}{\sin^2\varphi} - 1. \tag{42}$$

They are responsible for separability of H in the spherical coordinates. Moreover, it is not difficult to see that $\widetilde{L^2}$ and $\widetilde{L_z^2}$ are related to $C'^{\mathfrak{so}(5)}$ and $C'^{\mathfrak{so}(4)}$ in the sense that

$$\widetilde{L^2} = C'^{\mathfrak{so}(5)}|_{\mathfrak{H}_s}, \qquad \widetilde{L_z^2} = C'^{\mathfrak{so}(4)}|_{\mathfrak{H}_s} \tag{43}$$

where

$$C'^{\mathfrak{so}(5)} = (\sin\theta\sin\varphi)\circ C^{\mathfrak{so}(5)}\circ(\sin\theta\sin\varphi)^{-1},$$
$$C'^{\mathfrak{so}(4)} = (\sin\theta\sin\varphi)\circ C^{\mathfrak{so}(4)}\circ(\sin\theta\sin\varphi)^{-1}.$$

According to this the angular part $\mathcal{Y}_{\lambda ms}(\theta,\varphi)$ of wavefunctions is given by

$$\mathcal{Y}_{\lambda ms}(\theta,\varphi) = \chi\sin^{m+1}\theta\sin^{s+1}\varphi C_{\lambda-m}^{\frac{3}{2}+m}(\cos\theta)C_{m-s}^{1+s}(\cos\varphi), \tag{44}$$

with the normalization constant

$$(\chi)^2 = \frac{2^{1+2m+2s}(\lambda-m)!(m-s)!(s!)^2\Gamma^2\left(\frac{3}{2}+m\right)(3+2\lambda)(2+2m)}{\pi^2(\lambda+m+2)!(m+s+1)!}. \tag{45}$$

Here $C_n^{\lambda}(t)$ are the Gegenbauer polynomials [44].

Observe that the angle-function $\mathcal{Y}_{\lambda ms}(\theta,\varphi)$ depend on the details of the dynamics. This is a result of very general properties, shared by all non-central Hamiltonians. It is also worth noting that the functions $\mathcal{Y}_{\lambda ms}(\theta,\varphi)$ are related to matrix elements of class 1 representations of $\mathfrak{so}(5)$ [45] in the bases corresponding to $\mathfrak{so}(5)\supset\mathfrak{so}(4)\supset\cdots\supset\mathfrak{so}(2)$ reduction (see section 3).

3. Calculation of the *S* matrix

Once the group structure of the Schrödinger equation with potential (7) has been recognized, the associated S-matrix can be computed by using equation (4). This requires the use of matrices which intertwine Weyl-equivalent representations of $SO(5,1)$ or $\mathfrak{so}(5,1)$ in the bases corresponding to $SO(5,1) \supset SO(5) \supset \cdots \supset SO(2)$ reduction. We find it expedient to use, for this purpose, equation (5). By realizing the principal series of $SO(5,1)$ on suitable Hilbert spaces of some functions we can derive from equation (5) the functional relations for the kernel of A which allow us to obtain an integral representation for the matrix elements of A.

We shall start with the most degenerate principal series representations of $SO(5,1)$ associated with the cone [45]. These representations can be realized in the space of infinitely differentiable functions $f(\xi)$ on the upper sheet of the five-dimensional cone $\xi_1^2 + \cdots + \xi_5^2 - \xi_6^2 = 0$ $(\xi_6 > 0)$ that are homogeneous of degree $\sigma = -2 + \mathrm{i}\rho$

$$f(a\xi) = a^{\sigma} f(\xi), \qquad a > 0. \tag{46}$$

In this realization the group operators $U(g)$ are given by

$$U(g) f(\xi) = f(g^{-1}\xi). \tag{47}$$

The different choices of the coordinate system on the cone can lead to different subgroup reductions of $SO(5,1)$. The spherical coordinate system corresponding to the subgroup reduction $SO(5,1) \supset SO(5) \supset \cdots \supset SO(2)$ is given by

$$\xi = \omega\zeta, \qquad \zeta = (n, 1), \tag{48}$$

where $\omega = \xi_6$,

$$\begin{aligned}
n_1 &= \sin\theta_4 \sin\theta_3 \sin\theta_2 \sin\theta_1, \\
n_2 &= \sin\theta_4 \sin\theta_3 \sin\theta_2 \cos\theta_1, \\
n_3 &= \sin\theta_4 \sin\theta_3 \cos\theta_2, \\
n_4 &= \sin\theta_4 \cos\theta_3, \\
n_5 &= \cos\theta_4,
\end{aligned}$$

and

$$\mathrm{d}n = \sin^3\theta_4 \sin^2\theta_3 \sin\theta_2 \,\mathrm{d}\theta_4\, \mathrm{d}\theta_3\, \mathrm{d}\theta_2\, \mathrm{d}\theta_1. \tag{49}$$

From equation (46) it follows that the homogeneous function is defined uniquely by its values on the four-dimensional sphere S^4. Consequently, the most degenerate principal series representations of $SO(5,1)$ can be realized on $\mathcal{L}_2(S^4)$

$$U_\sigma(g) f(n) = (\omega_g)^{\sigma} f(n_g), \tag{50}$$

where ω_g and n_g are defined from

$$g^{-1}\zeta = \omega_g \zeta_g, \tag{51}$$

where $\zeta \equiv (n, 1)$ and $\zeta_g \equiv (n_g, 1)$. The operator A defined by

$$A f(n) = \int \mathcal{K}(n, n') f(n')\, \mathrm{d}n' \tag{52}$$

intertwines representations σ and $-4 - \sigma$, i.e.

$$A U_\sigma(g) = U_{-4-\sigma}(g) A \tag{53}$$

if

$$\mathcal{K}(n_g, n'_g) = (\omega_g)^{2+\sigma} (\omega'_g)^{2+\sigma} \mathcal{K}(n, n'). \tag{54}$$

In deriving equation (54), we have used the relation

$$\mathrm{d}n_g = (\omega_g)^{-4}\,\mathrm{d}n. \tag{55}$$

The kernel, $\mathcal{K}$, is uniquely determined by equation (54) up to a constant and is given by

$$\mathcal{K}(n, n') = \eta(1 - n \cdot n')^{-4-\sigma}. \tag{56}$$

The verification of equation (56) is based on the relation

$$1 - n_g \cdot n'_g = (\omega_g)^{-1}(\omega'_g)^{-1}(1 - n \cdot n'), \tag{57}$$

which obviously is a consequence of the relation

$$[g^{-1}\zeta, g^{-1}\zeta'] = [\zeta, \zeta']. \tag{58}$$

For the scattering systems under consideration we put

$$\eta = 2^{-2+\mathrm{i}\rho}\frac{\Gamma(2+\mathrm{i}\rho)}{\pi^2\Gamma(-\mathrm{i}\rho)}.$$

With this factor the operator A becomes unitary for $\sigma = -2+\mathrm{i}\rho$ (see equation (63)). Moreover, for $\gamma < 0$ (i.e. for an attractive Coulomb potential) it produces poles which correspond to bound states (see equation (68)).

Taking into account the fact that 4-dimensional spherical harmonics of degree λ, $Y_{\lambda K}$ [45]

$$Y_{\lambda K}(n) = \varkappa \mathrm{e}^{\mathrm{i}k_3\theta_1}\prod_{j=0}^{2} C_{k_j-k_{j+1}}^{k_{j+1}+(3-j)/2}(\cos\theta_{4-j})\sin^{k_{j+1}}\theta_{4-j}, \tag{59}$$

with

$$(\varkappa)^2 = \frac{1}{2\pi}\prod_{j=0}^{2}\frac{2^{2k_{j+1}+1-j}(k_j - k_{j+1})!(3 - j + 2k_j)\Gamma\left(\frac{3-j}{2} + k_{j+1}\right)}{\pi\Gamma(k_j + k_{j+1} + 3 - j)}$$

forms bases in $\mathcal{L}_2(S^4)$, corresponding to $SO(5,1) \supset SO(5) \supset \cdots \supset SO(2)$ reduction, we have the following integral representation for the matrix elements of A

$$\langle\lambda' K'|A|\lambda K\rangle = \int \mathcal{K}(n, n')Y_{\lambda K}(n')Y^*_{\lambda' K'}(n)\,\mathrm{d}n\,\mathrm{d}n', \tag{60}$$

where $\lambda \equiv k_0 \geqslant k_1 \geqslant k_2 \geqslant |k_3|$ and the symbol K denotes the sequence (k_1, k_2, k_3). The numbers k_i are all integers.

By using the expansion

$$(1 - n \cdot n')^{-2-\mathrm{i}\rho} = \sum_{\nu=0}^{\infty} b_\nu C_\nu^{3/2}(n \cdot n'), \tag{61}$$

where

$$\begin{aligned} b_\nu &= \frac{\nu!(2\nu+3)}{2\Gamma(3+\nu)}\int_0^\pi (1-\cos\theta)^{-2-\mathrm{i}\rho}C_\nu^{3/2}(\cos\theta)\sin^3\theta\,\mathrm{d}\theta \\ &= 2^{-1-\mathrm{i}\rho}\frac{\Gamma(-\mathrm{i}\rho)(2\nu+3)}{\Gamma(2+\mathrm{i}\rho)}\frac{\Gamma(2+\mathrm{i}\rho+\nu)}{\Gamma(2-\mathrm{i}\rho+\nu)} \end{aligned} \tag{62}$$

we have

$$\langle\lambda' K'|A|\lambda K\rangle = A_\lambda \delta_{\lambda\lambda'}\delta_{KK'} \tag{63}$$

with

$$A_\lambda = \frac{\Gamma(2+\mathrm{i}\rho+\lambda)}{\Gamma(2-\mathrm{i}\rho+\lambda)}.$$

In arriving at equation (63) we have used the addition formula

$$C_\nu^{3/2}(n \cdot n') = \frac{8\pi^2}{2\nu+3} \sum_K Y_{\nu K}(n) Y_{\nu K}^*(n') \tag{64}$$

and the orthogonality relation

$$\int \mathrm{d}n\, Y_{\nu K}(n) Y_{\nu' K'}^*(n) = \delta_{\nu\nu'}\delta_{KK'}. \tag{65}$$

Once the matrix elements of A have been obtained the S-matrix can be computed by using of equation (4). According to (63) we have

$$S(\theta,\varphi;\theta',\varphi') = \sum_{\lambda m} A_\lambda \mathcal{Y}_{\lambda m s}(\theta,\varphi)\mathcal{Y}_{\lambda m s}^*(\theta',\varphi'). \tag{66}$$

Then the partial wave expansion of the scattering amplitude $f(\theta,\varphi;\theta',\varphi')$ is defined by [46]

$$f(\theta,\varphi;\theta',\varphi') = \frac{2\pi}{\mathrm{i}p} \sum_{\lambda m} (A_\lambda - 1)\mathcal{Y}_{\lambda m s}(\theta,\varphi)\mathcal{Y}_{\lambda m s}^*(\theta',\varphi'). \tag{67}$$

Since

$$\sum_{\lambda m} \mathcal{Y}_{\lambda m s}(\theta,\varphi)\mathcal{Y}_{\lambda m s}^*(\theta',\varphi') = \delta(\cos\theta - \cos\theta')\delta(\varphi-\varphi')$$

we can omit unity in the bracket in (67) when $\theta \neq \theta'$, $\varphi \neq \varphi'$, leaving

$$f(\theta,\varphi;\theta',\varphi') = \frac{2\pi}{\mathrm{i}p} \sum_{\lambda m} A_\lambda \mathcal{Y}_{\lambda m s}(\theta,\varphi)\mathcal{Y}_{\lambda m s}^*(\theta',\varphi'). \tag{68}$$

The sum in (68) may be evaluated as follows. We start with the formula

$$\eta(1 - n\cdot n')^{-2-\mathrm{i}\rho} = \sum_{\lambda K} A_\lambda Y_{\lambda K}(n) Y_{\lambda K}^*(n') \tag{69}$$

with

$$Y_{\lambda K}(n) = (\sin\theta_4 \sin\theta_3)^{-1}\mathcal{Y}_{\lambda k_1 k_2}(\theta_4,\theta_3) Y_{k_2 k_3}(\theta_2,\theta_1) \tag{70}$$

where $Y_{k_2k_3}(\theta_2,\theta_1)$ is ordinary (2-dimensional) spherical harmonics of degree k_2. Putting $\theta_2' = \theta_1' = 0$ and using

$$Y_{k_2,k_3}(0,0) = \delta_{k_3,0}\left(\frac{2k_2+1}{4\pi}\right)^{1/2}, \qquad Y_{k_2,0}(\theta_2,\theta_1) = \left(\frac{2k_2+1}{4\pi}\right)^{1/2} P_{k_2}(\cos\theta_2)$$

we get

$$\begin{aligned}\eta[1 &- \sin\theta_4\sin\theta_4'\sin\theta_3\sin\theta_3'\cos\theta_2 - \sin\theta_4\sin\theta_4'\cos\theta_3\cos\theta_3' - \cos\theta_4\cos\theta_4']^{-2-\mathrm{i}\rho} \\ &= \frac{2k_2+1}{4\pi}\frac{1}{\sin\theta_4\sin\theta_4'\sin\theta_3\sin\theta_3'} \sum_{\lambda k_1 k_2} A_\lambda \mathcal{Y}_{\lambda k_1 k_2}(\theta_4,\theta_3)\mathcal{Y}_{\lambda k_1 k_2}(\theta_4',\theta_3') \\ &\quad \times P_{k_2}(\cos\theta_2),\end{aligned} \tag{71}$$

where P_k are Legendre polynomials. We multiply both sides of (71) by $P_k(\cos\theta_2)\sin\theta_2$ and integrate with respect to θ_2 from 0 to π. Taking into consideration the orthogonality relations for Legendre polynomials we obtain an integral representation for sum in (68). Hence, the following integral representation for the scattering amplitude holds

$$f(\theta,\varphi;\theta',\varphi') = \frac{2\pi}{\mathrm{i}p}\eta b \int_0^\pi (a - b\cos\alpha)^{-2-\mathrm{i}\rho} P_s(\cos\alpha)\sin\alpha\,\mathrm{d}\alpha \tag{72}$$

where

$$a = 1 - \sin\theta \sin\theta' \cos\varphi \cos\varphi' - \cos\theta \cos\theta' \tag{73}$$

and

$$b = \sin\theta \sin\theta' \sin\varphi \sin\varphi'. \tag{74}$$

We shall now ascertain the connection of the amplitude (72) with the associated Legendre functions. Putting

$$\cosh\delta = \frac{a}{\sqrt{a^2 - b^2}}, \qquad \sinh\delta = \frac{b}{\sqrt{a^2 - b^2}} \tag{75}$$

we have

$$f(\theta, \varphi; \theta', \varphi') = \frac{(2\pi)^2}{\mathrm{i}p} \eta b |a^2 - b^2|^{-\frac{2+\mathrm{i}\rho}{2}} \int_0^\pi (\cosh\delta - \sinh\delta \cos\alpha)^{-2-\mathrm{i}\rho} \times P_s(\cos\alpha) \sin\alpha \, \mathrm{d}\alpha. \tag{76}$$

On comparing this formula with formula 10.3.7(1) of [45], we obtain

$$f(\theta, \varphi; \theta', \varphi') = \frac{(2\pi)^{3/2}}{\mathrm{i}p} \eta b |a^2 - b^2|^{-\frac{2+\mathrm{i}\rho}{2}} \frac{\Gamma(1 + \mathrm{i}\rho)}{\Gamma(-1 - \mathrm{i}\rho - s)} \frac{1}{\sqrt{\sinh\delta}} P^{-\frac{1}{2}-s}_{-\frac{3}{2}-\mathrm{i}\rho}(\cosh\delta). \tag{77}$$

It is known that $P_\nu^{\pm(k+1/2)}$ in case $k = 0, 1, 2, \ldots$ reduce to a finite number of terms [47]. In particular we have

$$P_\nu^{-1/2}(z) = \left(\frac{2}{\pi}\right)^{1/2} \frac{(z^2 - 1)}{2\nu + 3} \{[z + (z^2 - 1)^{1/2}]^{\nu+1/2} - [z + (z^2 - 1)^{1/2}]^{\nu-1/2}\}$$

Therefore

$$f(\theta, \varphi; \theta', \varphi') = \frac{2^{\mathrm{i}\rho}}{\mathrm{i}p} \frac{\Gamma(1 + \mathrm{i}\rho)}{\Gamma(-\mathrm{i}\rho)} [(a - b)^{-1-\mathrm{i}\rho} - (a + b)^{-1-\mathrm{i}\rho}] \tag{78}$$

when $s = 0$, where

$$a \pm b = 1 - \sin\theta \sin\theta' \cos(\varphi \pm \varphi') - \cos\theta \cos\theta'$$

(also see the appendix).

We see that the amplitude (77) does not reduce to the Coulomb amplitude [48]

$$f_{\mathrm{Coul}}(\theta, \varphi; \theta', \varphi') = \frac{2^{\mathrm{i}\rho}}{\mathrm{i}p} \frac{\Gamma(1 + \mathrm{i}\rho)}{\Gamma(-\mathrm{i}\rho)} [1 - \sin\theta \sin\theta' \cos(\varphi - \varphi') - \cos\theta \cos\theta']^{-1-\mathrm{i}\rho} \tag{79}$$

when s is set equal to zero. The reason for this discrepancy lies in the fact that the Schrödinger equation with the potential (7) is supplemented with the boundary condition on the wavefunction at $\varphi = \pi$.

To make this point more precise we reexamine the angular part of the wavefunction for $s = 0$; using equation

$$C_m^1(\cos\varphi) = \frac{\sin(m + 1)\varphi}{\sin\varphi}$$

we find from (44) that

$$\mathcal{Y}_{\lambda m 0}(\theta, \varphi) = \chi_0 \sin^{m+1}\theta \, C^{\frac{3}{2}+m}_{\lambda-m}(\cos\theta) \sin(m + 1)\varphi \tag{80}$$

with

$$(\chi_0)^2 = (3 + 2\lambda) \frac{2^{2+2m} \Gamma^2(m + 3/2)(\lambda - m)!}{\pi^2 (\lambda + m + 2)!}. \tag{81}$$

Thus when $s = 0$, the angular part of the wavefunction is

$$\mathcal{Y}_{\lambda m 0}(\theta, \varphi) = \frac{1}{\mathrm{i}} [Y_{\lambda+1, m+1}(\theta, \varphi) - Y_{\lambda+1, m+1}(\theta, -\varphi)]$$

where we have used the well-known relation between Legendre and Gegenbauer functions. Hence, instead of the expression (79) for the scattering amplitude, we have (78).

Acknowledgment

I wish to thank A Ventura for his helpful discussions.

Appendix. An alternative proof for (78)

In this appendix, we give a direct calculation of the scattering amplitude (68) for $s = 0$

$$f(\theta,\varphi;\theta',\varphi') = \frac{2\pi}{\mathrm{i}p}\sum_{\lambda m} A_\lambda(\chi_0)^2 \sin^{m+1}\theta C_{\lambda-m}^{\frac{3}{2}+m}(\cos\theta)\sin^{m+1}\theta' C_{\lambda-m}^{\frac{3}{2}+m}(\cos\theta') \times \sin(m+1)\varphi\sin(m+1)\varphi', \tag{A.1}$$

where χ_0 is given by (81). The summation over m can be performed as follows. We start with the formula 9.4.2(3) of [45]

$$C_l^\nu(\cos\Omega) = \frac{\Gamma(2\nu-1)}{[\Gamma(\nu)]^2}\sum_{\mu=0}^{l}\frac{2^{2\mu}\Gamma^2(\nu+\mu)(l-\mu)!(2\mu+2\nu-1)}{\Gamma(l+\mu+2\nu)}\sin^\mu\theta C_{l-\mu}^{\nu+\mu}(\cos\theta) \times \sin^\mu\theta' C_{l-\mu}^{\nu+\mu}(\cos\theta')C_\mu^{\nu-1/2}(\cos\phi) \tag{A.2}$$

where $\cos\Omega = .\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi$. If we take the limit for $\nu \to 1/2$ and use

$$\lim_{p\to 0}\Gamma(p)C_n^p(\cos\phi) = \frac{2\cos n\phi}{n}$$

we get the equality

$$C_l^{1/2}(\cos\Omega) = \frac{2}{\pi}\sum_{\mu=0}^{l}\frac{2^{2\mu}\Gamma^2(1/2+\mu)(l-\mu)!}{\Gamma(l+\mu+1)}\sin^\mu\theta C_{l-\mu}^{1/2+\mu}(\cos\theta) \times \sin^\mu\theta' C_{l-\mu}^{1/2+\mu}(\cos\theta')\cos\mu\phi. \tag{A.3}$$

Further, putting $m + 1 = \mu$ and $\lambda + 1 = l$ in (A.1) and taking into account formula (A.3), we find from (A.1) that

$$f(\theta,\varphi;\theta',\varphi') = \frac{1}{2\mathrm{i}p}\sum_l (2l+1)\frac{\Gamma(1+\mathrm{i}\rho+l)}{\Gamma(1-\mathrm{i}\rho+l)}\left[C_l^{1/2}(\cos\Omega_-) - C_l^{1/2}(\cos\Omega_+)\right] \tag{A.4}$$

where $\cos\Omega_\pm = .\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\cos(\varphi\pm\varphi')$.

If we use the relation

$$\sum_l (2l+1)\frac{\Gamma(1+\mathrm{i}\rho+l)}{\Gamma(1-\mathrm{i}\rho+l)}C_l^{1/2}(z) = 2^{\mathrm{i}\rho+1}\frac{\Gamma(1+\mathrm{i}\rho)}{\Gamma(-\mathrm{i}\rho)}(1-z)^{-1-\mathrm{i}\rho} \tag{A.5}$$

in equation (A.4), we get the result obtained above (78).

References

[1] Fock V 1935 *Z. Phys.* **98** 145
[2] Makarov A A, Smorodinsky J A, Valiev Kh and Winternitz P 1967 *Nuovo Cimento* A **52** 1061
[3] Jauch J M 1939 *Phys. Rev.* A **55** 1132
[4] Hartmann H 1972 *Theor. Chim. Acta* **24** 201
[5] Sökmen I 1986 *Phys. Lett.* A **115** 249
[6] Chetonani L, Guechi L and Hamman T F 1987 *Phys. Lett.* A **125** 277
[7] Sökmen I 1988 *Phys. Lett.* A **132** 65
[8] Carpio-Bernido M V and Bernido C C 1989 *Phys. Lett.* A **134** 395
[9] Carpio-Bernido M V 1991 *J. Math. Phys.* **32** 1799

[10] Carpio-Bernido M V 1991 *J. Phys. A: Math. Gen.* **24** 3013
[11] Grosche C 1995 *Fortschr. Phys.* **43** 453
[12] Gravador E B, Carpio-Bernido M V and Bernido C C 1995 *Phys. Lett.* A **264** 45
[13] Khare A and Bhaduri R K 1994 *Am. J. Phys.* **62** 1008
[14] Gönül B and Zorba I 2000 *Phys. Lett.* A **269** 83
[15] Gerry C C 1986 *Phys. Lett.* A **118** 445
[16] Guha A and Mukherjee S 1987 *J. Math. Phys.* **28** 840
[17] Kibler M and Winternitz P 1987 *J. Phys. A: Math. Gen.* **20** 4097
[18] Quesne C 1988 *J. Phys. A: Math. Gen.* **21** 3093
[19] Corpio -Bernido M V and Bernido C C 1989 *Phys. Lett.* A **137** 1
[20] Vaidya A N and Boschi-Filho H 1990 *J. Math. Phys.* **31** 1951
[21] Boschi-Filho H and Vaidya A N 1990 *Phys. Lett.* A **149** 336
[22] Gramovskii Y I, Zhedanov A S and Lutzenko I M 1991 *J. Phys. A: Math. Gen.* **24** 3887
[23] Boschi-Filho H, De Sonza M and Vaidya A N 1991 *J. Phys. A: Math. Gen.* **24** 4981
[24] Gal'bert O F, Gramowski Y I and Zhedanow A S 1991 *Phys. Lett.* A **153** 179
[25] Chetonani L, Grechi L and Hamman T F 1992 *J. Math. Phys.* **33** 3410
[26] Zhedanov A S 1993 *J. Phys. A: Math. Gen.* **26** 4633
[27] Kerimov G A 2006 *J. Phys. A: Math. Gen.* **39** 1183
[28] Hartmann H and Schuch D 1980 *Int. J. Quantum Chem.* **18** 125
[29] Chen C Y, Liu C L and Sun D S 2002 *Phys. Lett.* A **305** 341
[30] Chen C Y, Sun D S and Liu C L 2003 *Phys. Lett.* A **317** 80
[31] Chen C Y, Lu F L and Sun D S 2004 *Phys. Lett.* A **325** 420
[32] Chen C Y, Sun D S and Lu F L 2006 *Phys. Lett.* A **348** 153
[33] Kerimov G A 1998 *Phys. Rev. Lett.* **80** 2976
[34] Alhassid Y, Gursey F and Iachello F 1983 *Ann. Phys., NY* **148** 346
[35] Kerimov G A and Sezgin M 1998 *J. Phys. A: Math. Gen.* **31** 7901
[36] Baran S A, Dalgic A and Kerimov G A 2001 *J. Math. Phys.* **42** 543
[37] Kerimov G A 2002 *Phys. Lett.* A **294** 278
[38] Baran S A and Kerimov G A 2004 *J. Math. Phys.* **45** 1249
[39] Kerimov G A 2005 *J. Phys. A: Math. Gen.* **38** 1931
[40] Kerimov G A and Ventura A 2006 *J. Math. Phys.* **45** 082108
[41] Zwanziger D 1968 *Phys. Rev.* **176** 1480
[42] McIntosh H and Cisneros A 1970 *J. Math. Phys.* **11** 896
[43] Rasmussen W O and Salamo S 1979 *J. Math. Phys.* **20** 543
[44] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series and Products* (New York: Academic)
[45] Vilenkin N Ya 1968 *Special Functions and the Theory of Group Representations* (Providence, RI: American Mathematical Society)
[46] Newton R G 1982 *Scattering Theory of Waves and Particles* (New York: Springer)
[47] Bateman H 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)
[48] Landau L D and Lifshitz E M 1977 *Quantum Mechanics* 3rd edn (Oxford: Pergamon)